The isotopy problem of braids
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N-KOOK Seminar, Osaka State University, May 16, 2015
The braid isotopy problem is a problem of medium difficulty,
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  ▶ two topological solutions: Dynnikov’s coordinates, Bressaud’s relaxation method
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Here: a survey of some solutions:

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  [and two more: the alternating normal form (yesterday),]
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• Here: a survey of some solutions:
  ▶ one algebraic solution: the greedy normal form
  ▶ two topological solutions: Dynnikov’s coordinates, Bressaud’s relaxation method
    [and two more: the alternating normal form (yesterday), handle reduction (ILD)]
Plan:
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- 1. The braid isotopy problem
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- 2. Greedy normal form and the Garside structure
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• 4. Bressaud’s relaxation algorithm
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The braid isotopy problem
• A 3-strand braid diagram:
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(no U-turn allowed)
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(no U-turn allowed)

• Isotopy Problem:
The braid isotopy problem

- A 3-strand braid diagram:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{braid_diagram.png}
\end{array}
\]

(no U-turn allowed)

- **Isotopy Problem:**
  Given two $n$-strand braid diagrams, can one deform them to one another?
The braid isotopy problem

- **A 3-strand braid diagram:**

  ![A 3-strand braid diagram]

  (no U-turn allowed)

- **Isotopy Problem:**
  Given two $n$-strand braid diagrams, can one deform them to one another?

  ![Isotopy Problem]
The braid isotopy problem

- A 3-strand braid diagram:

![3-strand braid diagram](image)

(no U-turn allowed)

- Isotopy Problem:
  Given two $n$-strand braid diagrams, can one deform them to one another?

![Deformed braid diagrams](image)

is isotopic to
• A 3-strand braid diagram:

![3-strand braid diagram](image)

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• Isotopy Problem:
  Given two \( n\)-strand braid diagrams, can one deform them to one another?

![Isotopy example](image)

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\[
\text{is isotopic to}
\]
The braid isotopy problem

• A 3-strand braid diagram:

• Isotopy Problem: Given two $n$-strand braid diagrams, can one deform them to one another?

• More formally: view braid diagrams as projections of 3D-diagrams in $D^2 \times (0, 1)$,
The braid isotopy problem

- A 3-strand braid diagram:

![Braid diagram with no U-turn allowed](image)

- Isotopy Problem:
  Given two \( n \)-strand braid diagrams, can one deform them to one another?

![Isotopic braid diagrams](image)

- More formally: view braid diagrams as projections of 3D-diagrams in \( D^2 \times (0, 1) \),

![Projection of 3D-diagram](image)
- A 3-strand braid diagram:

- Isotopy Problem: Given two $n$-strand braid diagrams, can one deform them to one another?

- More formally: view braid diagrams as projections of 3D-diagrams in $D^2 \times (0, 1)$,
The braid isotopy problem

• A 3-strand braid diagram:

![Braid Diagram](image)

(no U-turn allowed)

• Isotopy Problem:
  Given two $n$-strand braid diagrams, can one deform them to one another?

![Isotopy Example](image)

is isotopic to

• More formally: view braid diagrams as projections of 3D-diagrams in $D^2 \times (0,1)$,

![3D Diagram](image)

and consider ambient isotopy leaving the end-disks fixed.
• Concatenation of braid diagrams:
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- Associative;
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- Compatible with isotopy, hence induces a well-defined product on classes;
• **Concatenation** of braid diagrams:

![Example of concatenation of braid diagrams]

- Associative;
- Compatible with isotopy, hence induces a well-defined product on classes;
- Admits the unbraided diagram $[\emptyset]$ as a neutral element;
• **Concatenation** of braid diagrams:

![Diagram]

- Associative;
- Compatible with isotopy, hence induces a well-defined product on classes;
- Admits the unbraided diagram $[\emptyset]$ as a neutral element;
- Every diagram has an inverse, its `mirror`-image:
Braid groups

- **Concatenation** of braid diagrams:

  
  
  ![Braid Diagrams](image)

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  - Every diagram has an inverse, its mirror-image:

  ![Braid Diagram](image)
• **Concatenation** of braid diagrams:

\[
\begin{array}{c}
\text{braid} \\
\text{braid} \\
\text{braid}
\end{array}
\ast
\begin{array}{c}
\text{braid} \\
\text{braid} \\
\text{braid}
\end{array} =
\begin{array}{c}
\text{braid} \\
\text{braid} \\
\text{braid}
\end{array}
\]

- Associative;
- Compatible with isotopy, hence induces a well-defined product on classes;
- Admits the unbraided diagram \([\emptyset]\) as a neutral element;
- Every diagram has an inverse, its **mirror**-image:
• **Concatenation** of braid diagrams:

\[
\begin{array}{c}
\text{braid} \quad \ast \quad \text{braid} \\
\text{braid} \quad \ast \quad \text{braid}
\end{array}
\]

- Associative;
- Compatible with isotopy, hence induces a well-defined product on classes;
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\[ \begin{array}{ccc}
\text{braid} & \ast & \text{braid} \\
\text{braid} & \ast & \text{braid}
\end{array} \]

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- For every $n \geq 1$: the group $B_n$ of $n$-strand braids.
- **Concatenation** of braid diagrams:

  [Diagram showing concatenation of braid diagrams]

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  [Diagram showing the mirror-image]

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• The group structure of $B_n$ makes the Braid Isotopy Problem easier:
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The group structure of $B_n$ makes the Braid Isotopy Problem easier:

- Reduces to the Braid Triviality Problem: $D' \approx D \Leftrightarrow D^{-1} \ast D' \approx [\emptyset]$.
- Enables one to use algebraic tools,
Artin’s presentation

• The group structure of $B_n$ makes the Braid Isotopy Problem easier:
  ▶ Reduces to the Braid Triviality Problem: $D' \cong D \iff D^{-1} \ast D' \cong [\emptyset]$.
  ▶ Enables one to use algebraic tools, provided one has a presentation of $B_n$. 
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• **Artin** generators: Every $n$-strand braid diagram is a (finite) concatenation of elementary diagrams with one crossing,
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• Artin generators: Every $n$-strand braid diagram is a (finite) concatenation of elementary diagrams with one crossing, hence of the form

\[
\sigma_i : \begin{array}{c}
\vdots \\
\vdots \\
\sigma_i \\
\vdots \\
\vdots \\
\end{array} \begin{array}{c}
\vdots \\
\vdots \\
i+1 \\
i \\
\vdots \\
1 \\
\end{array}
\]
• The group structure of $B_n$ makes the Braid Isotopy Problem easier:
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\vdots \\
\vdots \\
\vdots \\
\begin{array}{c}
\vdots \\
\vdots \\
i+1
\vdots \\
\vdots \\
\vdots \\
\vdots \\
n
\end{array}
\end{array}$

or

$\sigma_i^{-1} : \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\begin{array}{c}
\vdots \\
\vdots \\
i+1
\vdots \\
\vdots \\
\vdots \\
\vdots \\
n
\end{array}
\end{array}$

with $1 \leq i < n$. 


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\[
\sigma_i : \begin{array}{c}
\vdots \\
\vdots \\
1 \quad i+1 \\
\vdots \\
\vdots 
\end{array}
\quad \text{or} \quad \sigma_i^{-1} : \begin{array}{c}
\vdots \\
\vdots \\
1 \quad i+1 \\
\vdots \\
\vdots 
\end{array}
\]

with $1 \leq i < n$.

• Theorem (Artin, 1926): The group $B_n$ admits the presentation

\[\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2 \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1 \rangle.\]
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\vdots \\
\vdots \\
\vdots \\
1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
i+1 \\
i \\
i \\
i+1
\end{array}
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\vdots \\
1 \\
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\vdots \\
\vdots \\
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i+1 \\
i \\
i \\
i+1
\end{array}
$$

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• **Theorem** (Artin, 1926): The group $B_n$ admits the presentation

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\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{for } |i - j| = 1 \rangle.
$$

▶ Proof: Isotopy of piecewise linear diagrams is generated by $\Delta$-moves. □
Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
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• Braid Isotopy reduced to the Word Problem for $B_n$ with respect to $\{\sigma_1, \ldots, \sigma_{n-1}\}$:
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  a word in the letters $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$.
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• (Novikov, 1952) There exists a finitely presented group with an unsolvable Word Problem.
Reducing to the monoid

- Braid Isotopy reduced to the Word Problem for $B_n$ with respect to \(\{\sigma_1, \ldots, \sigma_{n-1}\}\): given a braid word $w$, decide whether $w$ represents 1 in $B_n$.

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- Here: $(\text{Garside})$ Use the monoid.
Reducing to the monoid

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- **Theorem** (Garside, 1969): Let $B_n^+$ be the monoid with presentation

  \[
  \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \begin{align*}
  \sigma_i \sigma_j &= \sigma_j \sigma_i & \text{for } |i - j| \geq 2 \\
  \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1
  \end{align*} \right\rangle^+.
  \]
• Braid Isotopy reduced to the **Word Problem** for $B_n$ with respect to $\{\sigma_1, ..., \sigma_{n-1}\}$:
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  $$\left\langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| \geq 2 \right\rangle^+.$$  

  Then $B_n^+$ embeds in $B_n$ and $B_n$ is a **group of fractions** for $B_n^+$.

  every element of $B_n$ can be written $\beta^{-1} \gamma$ with $\beta, \gamma \in B_n^+$
Reducing to the monoid

- Braid Isotopy reduced to the **Word Problem** for $B_n$ with respect to \{\sigma_1, \ldots, \sigma_{n-1}\}:
  
  given a **braid word** $w$, decide whether $w$ represents 1 in $B_n$.

- \begin{align*}
  & \text{a word in the letters } \sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}.
\end{align*}

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Then $B_n^+$ embeds in $B_n$ and $B_n$ is a *group of fractions* for $B_n^+$.

\begin{itemize}
  \item every element of $B_n$ can be written $\beta^{-1} \gamma$ with $\beta, \gamma \in B_n^+$
  \item Proof: Show that $B_n^+$ is cancellative and admits common multiples.
\end{itemize}
• An **effective** way of reducing from $B_n$ to $B_n^+$:
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• **Lemma** (Garside): *Inductively define* $\Delta_n$ *by* $\Delta_1 = 1$, $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1$.
• An effective way of reducing from $B_n$ to $B_n^+$:

• **Lemma** (Garside): *Inductively define $\Delta_n$ by $\Delta_1 = 1$, $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1$.*

Then, for every (signed) $n$-strand braid word $w$, one can find $p \geq 0$ and a positive $n$-strand braid word $w'$ and satisfying $\Delta_n^p w \equiv w'$.
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  \[
  \begin{array}{c}
  \text{\includegraphics[width=2cm]{braids.png}}
  \\
  \cong \text{\includegraphics[width=2cm]{braids2.png}}
  \end{array}
  
  
  
  Then, for every (signed) $n$-strand braid word $w$, one can find $p \geq 0$

  and a positive $n$-strand braid word $w'$ and satisfying $\Delta_n^p w \equiv w'$.

  
  
  
  • Then: \( w \equiv \in \uparrow \)

  \[
  \text{the empty word}
  \]
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• Then: $w \equiv \varepsilon \iff w' \equiv \Delta_n^p$

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\[
\begin{array}{c}
\text{Then, for every (signed) } n\text{-strand braid word } w, \text{ one can find } p \geq 0 \\
\text{and a positive } n\text{-strand braid word } w' \text{ and satisfying } \Delta_n^p w \equiv w'.
\end{array}
\]

• Then: 
  \[
  w \equiv \varepsilon \iff w' \equiv \Delta_n^p \iff w' \equiv^+ \Delta_n^p
  \]
  
  \[
  \begin{array}{c}
  \text{the empty word} \quad \text{equivalence} \\
  \text{generated by braid relations} \\
  \text{and } \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1
  \end{array}
  \]
An effective way of reducing from $B_n$ to $B_n^+$:

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↑ the empty word equivalence equivalence

↑ generated by braid relations generated by braid relations alone

and $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$
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the empty word equivalence equivalence

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Now: $\equiv^+$ is decidable, as it preserves word-length.
An effective way of reducing from $B_n$ to $B_n^+$:

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the empty word  

\[
\equiv \quad \sim \quad \equiv^+ \quad \sim
\]

equivalence  

\[
\text{equivalence generated by braid relations}
\quad \text{equivalence generated by braid relations alone}
\]

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Hence: A (theoretical) solution to the Braid Isotopy Problem: starting from $w$, 

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\[
\begin{align*}
\text{Then, for every (signed) } n\text{-strand braid word } w, & \text{ one can find } p \geq 0 \\
& \text{and a positive } n\text{-strand braid word } w' \text{ and satisfying } \Delta_n^p w \equiv w'.
\end{align*}
\]

• Then: $w \equiv \varepsilon \iff w' \equiv \Delta_n^p \iff w' \equiv^+ \Delta_n^p$

  the empty word \hspace{1cm} equivalence \hspace{1cm} equivalence

  generated by braid relations \hspace{1cm} generated by braid relations alone

  and $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$

• Now: $\equiv^+$ is decidable, as it preserves word-length.

• Hence: A (theoretical) solution to the Braid Isotopy Problem: starting from $w$,
  
  1. find $p$ and $w'$ positive satisfying $\Delta_n^p w \equiv w'$;
• An effective way of reducing from $B_n$ to $B_n^+$:

• **Lemma** (Garside): *Inductively define $\Delta_n$ by $\Delta_1 = 1$, $\Delta_n = \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1$.*

\[
\begin{align*}
\Delta_1 &= 1, \\
\Delta_n &= \Delta_{n-1} \cdot \sigma_{n-1} \cdots \sigma_2 \sigma_1.
\end{align*}
\]

Then, for every (signed) $n$-strand braid word $w$, one can find $p \geq 0$ and a positive $n$-strand braid word $w'$ and satisfying $\Delta_n^p w \equiv w'$.

• Then: $w \equiv \varepsilon \quad \Leftrightarrow \quad w' \equiv \Delta_n^p \quad \Leftrightarrow \quad w' \equiv^+ \Delta_n^p$

\[
\begin{align*}
\uparrow & \quad \text{the empty word} \quad \uparrow & \quad \text{equivalence} \quad \uparrow & \quad \text{equivalence} \\
& \quad \text{generated by braid relations} \quad & \quad \text{generated by braid relations alone} \\
& \quad \text{and } \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1
\end{align*}
\]

• Now: $\equiv^+$ is decidable, as it preserves word-length.

• Hence: A (theoretical) solution to the Braid Isotopy Problem: starting from $w$,
  1. find $p$ and $w'$ positive satisfying $\Delta_n^p w \equiv w'$;
  2. test $w' \equiv^+ \Delta_n^p$ by systematically enumerating the $\equiv^+$-class of $w'$. 
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   a family of $n!$ permutation braids in $B_n^+$.

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  $\beta$ left-divides $\gamma$ if $\exists \gamma' \ (\beta \gamma' = \gamma)$. 
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\[
\begin{array}{c}
\Delta_n^p \\
\sigma_{f_1}
\end{array}
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\[
\begin{align*}
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  ► The normal form of $\sigma_g \beta$ is $\Delta_n^p \sigma_{f_1'} \cdots \sigma_{f_p'} \sigma_{g_p}$ if $\sigma_{f_1'} \neq \Delta_n$. 


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- Assume that the normal form of $\beta$ is $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

\[
\begin{align*}
\sigma_g & \quad \rightarrow \quad \sigma_{g_0} \\
\Delta_n^p & \quad \rightarrow \quad \sigma_{f_1} \\
\sigma_{g_0} & \quad \rightarrow \quad \sigma_{g_1} \\
\sigma_{f_1} & \quad \rightarrow \quad \sigma_{f_2} \\
\sigma_{g_1} & \quad \rightarrow \quad \sigma_{g_2} \\
\ldots & \quad \rightarrow \quad \ldots \\
\sigma_{g_{r-1}} & \quad \rightarrow \quad \sigma_{g_r} \\
\sigma_{f_r} & \quad \rightarrow \quad \Delta_n^{p+1} \sigma_{f_1} \cdots \sigma_{f_p} \sigma_{g_p}
\end{align*}
\]

- The normal form of $\sigma_g \beta$ is $\Delta_n^p \sigma_{f_1}' \cdots \sigma_{f_p}' \sigma_{g_p}$ if $\sigma_{f_1}' \neq \Delta_n$,
  and $\Delta_n^{p+1} \sigma_{f_1}' \cdots \sigma_{f_p}' \sigma_{g_p}$ otherwise.
The greedy normal form (cont’d)

• The point here: not only theoretical, but also tractable.
  ▶ The greedy normal form can be computed efficiently.
  ▶ Key point: computing the normal form of $\sigma_i\beta$ and $\sigma_i^{-1}\beta$ from that of $\beta$.

• Recipe:
  ▶ Assume that the normal form of $\beta$ is $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

\[
\begin{align*}
\Delta_n^p & \quad \Delta_n^p \\
\sigma_g & \quad \sigma_{\phi_n^p(g)} & \quad \sigma_{f_1} & \quad \sigma_{f_2} & \quad \cdots & \quad \sigma_{f_r} & \quad \sigma_{g_{r-1}} & \quad \sigma_{g_r} \\
& \quad \sigma_{g_0} & \quad \sigma_{g_1} & \quad \sigma_{g_2} \\
\end{align*}
\]

▶ The normal form of $\sigma_g\beta$ is $\Delta_n^p \sigma_{f_1'} \cdots \sigma_{f_p'} \sigma_{g_p}$ if $\sigma_{f_1'} \neq \Delta_n$, and $\Delta_n^{p+1} \sigma_{f_2'} \cdots \sigma_{f_p'} \sigma_{g_p}$ otherwise.

▶ And the normal form of $\sigma_g^{-1}\beta$?
• The point here: not only theoretical, but also tractable.
  ▶ The greedy normal form can be computed efficiently.
  ▶ Key point: computing the normal form of $\sigma_i \beta$ and $\sigma_i^{-1} \beta$ from that of $\beta$.

• Recipe:
  ▶ Assume that the normal form of $\beta$ is $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

  $\Delta_n^p \sigma_g \xrightarrow{\phi_n(g)} \Delta_n^p \sigma_{f_1} \xrightarrow{\sigma_{f_1}} \Delta_n^p \sigma_{g_0} \xrightarrow{\sigma_{f_1}} \Delta_n^p \sigma_{f_2} \xrightarrow{\sigma_{f_2}} \cdots \xrightarrow{\sigma_{f_r}} \Delta_n^p \sigma_{g_r}$

  ▶ The normal form of $\sigma_g \beta$ is $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_p} \sigma_g \sigma_{g_0}$ if $\sigma_{f_1} \neq \Delta_n$,
    and $\Delta_n^{p+1} \sigma_{f_1} \cdots \sigma_{f_p} \sigma_g \sigma_{g_0}$ otherwise.

  ▶ And the normal form of $\sigma_g^{-1} \beta$? There exists $g'$ satisfying $\sigma_g \sigma_{g'} = \Delta_n$. 

The greedy normal form (cont’d)
• The point here: not only theoretical, but also tractable.
  ▶ The greedy normal form can be computed efficiently.
  ▶ Key point: computing the normal form of $\sigma_i \beta$ and $\sigma_i^{-1} \beta$ from that of $\beta$.

• Recipe:
  ▶ Assume that the normal form of $\beta$ is $\Delta_n^p \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

  $\Delta_n^p \xrightarrow{\Delta_n^p} \sigma_{f_1} \xrightarrow{\sigma_{f_1}'} \sigma_{f_2} \xrightarrow{\sigma_{f_2}'} \cdots \xrightarrow{\sigma_{f_r}'} \sigma_{g_r}$

  ▶ The normal form of $\sigma_g \beta$ is $\Delta_n^p \sigma_{f_1}' \cdots \sigma_{f_p}' \sigma_{g_p}$ if $\sigma_{f_1}' \neq \Delta_n$, and $\Delta_n^{p+1} \sigma_{f_2}' \cdots \sigma_{f_p}' \sigma_{g_p}$ otherwise.

  ▶ And the normal form of $\sigma_g^{-1} \beta$? There exists $g'$ satisfying $\sigma_g \sigma_{g'} = \Delta_n$, hence $\sigma_g^{-1} = \sigma_{g'} \Delta_n^{-1}$,
The point here: not only theoretical, but also tractable.
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Recipe:
  ▶ Assume that the normal form of $\beta$ is $\Delta^p_n \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

  $\begin{array}{c}
  \Delta^p_n \\
  \sigma_g \\
  \phi^p_n(g) \\
  \Delta^p_n \\
  \sigma_g \sigma_{f_1} \\
  \sigma_{f_1}' \\
  \sigma_{f_2} \\
  \sigma_{f_2}' \\
  \sigma_{f_r} \\
  \sigma_{f_r}' \\
  \sigma_g \sigma_{g_1} \\
  \sigma_g \sigma_{g_2} \\
  \sigma_g \sigma_{g_{r-1}} \\
  \sigma_g \sigma_{g_r}
  \end{array}$

  ▶ The normal form of $\sigma_g \beta$ is $\Delta^p_n \sigma_{f_1}' \cdots \sigma_{f_p}' \sigma_{g_p}$ if $\sigma_{f_1}' \neq \Delta_n$,
and $\Delta^{p+1}_n \sigma_{f_2}' \cdots \sigma_{f_p}' \sigma_{g_p}$ otherwise.

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hence $\sigma_g^{-1} = \sigma_{g'} \Delta_n^{-1}$, and $\sigma_g^{-1} \beta = \sigma_{g'} \Delta_n^{p-1} \sigma_{f_1} \cdots \sigma_{f_r}$:
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• Recipe:
  ▶ Assume that the normal form of $\beta$ is $\Delta_p^n \sigma_{f_1} \cdots \sigma_{f_r}$; let $\sigma_g$ be a permutation-braid;

\[
\begin{array}{ccccccc}
\sigma_g & \xrightarrow{\Delta_p^n} & \sigma_{\phi_p^n(g)} & \xrightarrow{\sigma_{f_1}'} & \sigma_{g_0} & \xrightarrow{\sigma_{f_2}'} & \sigma_{g_1} & \xrightarrow{\sigma_{g_2}} & \cdots & \cdots & \sigma_{g_{r-1}} & \xrightarrow{\sigma_{f_r}'} & \sigma_{g_r} \\
\sigma_g & \xrightarrow{\Delta_p^n} & \sigma_{f_1} & \sigma_{f_2} & \cdots & \cdots & \sigma_{f_{r-1}} & \sigma_{f_r} & \cdots & \cdots & \sigma_{f_{r-1}} & \sigma_{g_{r-1}} & \sigma_{g_r} \\
\end{array}
\]

▶ The normal form of $\sigma_g\beta$ is $\Delta_p^n \sigma_{f_1}' \cdots \sigma_{f_p}' \sigma_{g_p}$ if $\sigma_{f_1}' \neq \Delta_n$, and $\Delta_p^{n+1} \sigma_{f_2}' \cdots \sigma_{f_p}' \sigma_{g_p}$ otherwise.

▶ And the normal form of $\sigma_g^{-1}\beta$? There exists $g'$ satisfying $\sigma_g \sigma_{g'} = \Delta_n$, hence $\sigma_g^{-1} = \sigma_{g'} \Delta_n^{-1}$, and $\sigma_g^{-1} \beta = \sigma_{g'} \Delta_n^{p-1} \sigma_{f_1} \cdots \sigma_{f_r}$: continue as above.
• This corresponds to an automatic structure for $B_n$ (Thurston, Cannon),
• This corresponds to an automatic structure for $B_n$ (Thurston, Cannon), and, more specifically, to a Garside structure (D.–Paris 1997):
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
  ▶ and, more specifically, to a **Garside structure** (D.–Paris 1997):

  a submonoid $B_n^+$ of $B_n$. 

  

• This corresponds to an automatic structure for $B_n$ (Thurston, Cannon),
  ▶ and, more specifically, to a Garside structure (D.–Paris 1997):

  a submonoid $B_n^+$ of $B_n$, plus an element $\Delta_n$ of $B_n^+$ such that
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  - $B_n$ is a group of fractions for $B_n^+$,
  - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
  - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$,
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
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    - a submonoid $B_n^+$ of $B_n$, plus an element $\Delta_n$ of $B_n^+$ such that
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      - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$,
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  ► Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?
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Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

The dual Garside structure on $B_n$, 

Garside structures on $B_n$
This corresponds to an automatic structure for $B_n$ (Thurston, Cannon), and, more specifically, to a Garside structure (D.–Paris 1997):

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Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

The dual Garside structure on $B_n$, based on the Birman–Ko–Lee generators:
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
  - and, more specifically, to a **Garside structure** (D.–Paris 1997):
    
    a submonoid $B_n^+$ of $B_n$, plus an element $\Delta_n$ of $B_n^+$ such that
    - $B_n$ is a group of fractions for $B_n^+$,
    - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
    - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.
  - Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

• The **dual** Garside structure on $B_n$, based on the **Birman–Ko–Lee** generators:
  
  for $1 \leq i < j \leq n$: \[ a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}. \]
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
  - and, more specifically, to a **Garside structure** (D.–Paris 1997):

    ![Diagram of a submonoid $B^+_n$ of $B_n$, plus an element $\Delta_n$ of $B^+_n$ such that $B_n$ is a group of fractions for $B^+_n$, $B^+_n$ equipped with the (left) divisibility relation is a lattice, $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B^+_n$, and $\#\text{Div}(\Delta_n) < \infty$.]

  - Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

• The **dual** Garside structure on $B_n$, based on the Birman–Ko–Lee generators:
  - for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}$. 


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      - $B_n$ is a group of fractions for $B_n^+$,
      - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
      - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.

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• The **dual** Garside structure on $B_n$, based on the Birman–Ko–Lee generators:
  - for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$.

• **Definition** (Birman–Ko–Lee 1997): $B_n^{**} :=$ submonoid of $B_n$ generated by the $a_{i,j}$s.
• This corresponds to an automatic structure for $B_n$ (Thurston, Cannon),
  and, more specifically, to a Garside structure (D.–Paris 1997):
  - a submonoid $B_n^+$ of $B_n$, plus an element $\Delta_n$ of $B_n^+$ such that
    - $B_n$ is a group of fractions for $B_n^+$,
    - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
    - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.

• Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

• The dual Garside structure on $B_n$, based on the Birman–Ko–Lee generators:
  for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}$.

• Definition (Birman–Ko–Lee 1997): $B_n^{**} := \text{submonoid of } B_n \text{ generated by the } a_{i,j}$s.
  $\Delta_n^* := a_{1,2} a_{2,3} \cdots a_{n-1,n}$
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
  - and, more specifically, to a **Garside structure** (D.-Paris 1997):
    - a submonoid $B_n^+$ of $B_n$, plus an element $\Delta_n$ of $B_n^+$ such that
      - $B_n$ is a group of fractions for $B_n^+$,
      - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
      - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.
  - Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

• The **dual** Garside structure on $B_n$, based on the Birman–Ko–Lee generators:
  - for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1}$.

![Diagram](image)

• **Definition** (Birman–Ko–Lee 1997): $B_n^{++} :=$ submonoid of $B_n$ generated by the $a_{i,j}$s.
  $\Delta_n^* := a_{1,2} a_{2,3} \cdots a_{n-1,n} (= \sigma_1 \sigma_2 \cdots \sigma_{n-1})$. 
This corresponds to an automatic structure for $B_n$ (Thurston, Cannon), and, more specifically, to a Garside structure (D.–Paris 1997):

- $B_n$ is a group of fractions for $B_n^+$,
- $B_n^+$ equipped with the (left) divisibility relation is a lattice,
- $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.

Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

The dual Garside structure on $B_n$, based on the Birman–Ko–Lee generators:

for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$.

Definition (Birman–Ko–Lee 1997): $B_n^{++} :=$ submonoid of $B_n$ generated by the $a_{i,j}$s.

$\Delta_n^* := a_{1,2}a_{2,3} \cdots a_{n-1,n} (= \sigma_1 \sigma_2 \cdots \sigma_{n-1})$.

Proposition: $(B_n^{++}, \Delta_n^*)$ is a Garside structure on $B_n$. 
• This corresponds to an **automatic structure** for $B_n$ (Thurston, Cannon),
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    $B_n$ is a group of fractions for $B_n^+$,
    - $B_n^+$ equipped with the (left) divisibility relation is a lattice,
    - $\text{Div}_{\text{left}}(\Delta_n) = \text{Div}_{\text{right}}(\Delta_n)$, $\text{Div}(\Delta_n)$ generates $B_n^+$, and $\#\text{Div}(\Delta_n) < \infty$.

  ▶️ Is the Garside structure on $B_n$ unique? Is there another Garside structure on $B_n$?

• The **dual** Garside structure on $B_n$, based on the Birman–Ko–Lee generators:

  for $1 \leq i < j \leq n$: $a_{i,j} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$.

  ![Diagram](image)

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  ▶️ **Definition** (Birman–Ko–Lee 1997): $B_n^{++} := \text{submonoid of } B_n \text{ generated by the } a_{i,j} \text{s}.$
  $\Delta_n^* := a_{1,2} a_{2,3} \cdots a_{n-1,n} (= \sigma_1 \sigma_2 \cdots \sigma_{n-1})$.

  ▶️ **Proposition**: $(B_n^{++}, \Delta_n^*)$ is a **Garside structure** on $B_n$.
  - a new solution of the Word Problem.
Chord representation of the Birman–Ko–Lee generators:
- Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$
• Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$

• Lemma: In terms of the BKL generators, $B_n$ is presented by the relations
• Chord representation of the Birman–Ko–Lee generators: \[ a_{i,j} \mapsto \]

• **Lemma:** *In terms of the BKL generators, \( B_n \) is presented by the relations*

\[
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array} \\
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array} \\
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array}
\end{array}
\]

\[ = \]

\[
\begin{array}{c}
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array} \\
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array} \\
\begin{array}{ccc}
\circ & \circ & \circ \\
\hline
\circ & \circ & \circ \\
\end{array}
\end{array}
\]

\[ \text{for disjoint chords,} \]
• Chord representation of the Birman–Ko–Lee generators: $a_{i,j} \mapsto$.

• **Lemma:** *In terms of the BKL generators, $B_n$ is presented by the relations for disjoint chords,*

\[
\begin{align*}
\text{Diagram 1} & \cdot \text{Diagram 2} = \text{Diagram 3} \cdot \text{Diagram 4}
\end{align*}
\]
- Chord representation of the Birman–Ko–Lee generators: $a_{ij} \mapsto \begin{array}{c}
1 \\
j \\
i
\end{array}$

- **Lemma**: In terms of the BKL generators, $B_n$ is presented by the relations

\[
\begin{array}{c}
1 \\
j \\
i
\end{array} \cdot \begin{array}{c}
1 \\
j \\
i
\end{array} = \begin{array}{c}
1 \\
j \\
i
\end{array} \cdot \begin{array}{c}
1 \\
j \\
i
\end{array}
\]

\[
\begin{array}{c}
1 \\
j \\
i
\end{array} \cdot \begin{array}{c}
1 \\
j \\
i
\end{array} = \begin{array}{c}
1 \\
j \\
i
\end{array} \cdot \begin{array}{c}
1 \\
j \\
i
\end{array}
\]

for disjoint chords,
• Chord representation of the Birman–Ko–Lee generators: \( a_{i,j} \mapsto \)

• **Lemma:** *In terms of the BKL generators, \( B_n \) is presented by the relations*

\[
\begin{align*}
\text{for disjoint chords,} & \\
\text{for adjacent chords enumerated in clockwise order.}
\end{align*}
\]
• Chord representation of the Birman–Ko–Lee generators: \( a_{i,j} \mapsto \)

• Lemma: In terms of the BKL generators, \( B_n \) is presented by the relations

\[
\begin{align*}
\text{for disjoint chords}, \\
\text{for adjacent chords enumerated in clockwise order.}
\end{align*}
\]

Hence: For \( P \) a \( p \)-gon, can define \( a_P \) to be the product of the \( a_{i,j} \) corresponding to \( p-1 \) adjacent edges of \( P \) in clockwise order;
• Chord representation of the Birman–Ko–Lee generators: \( a_{i,j} \)

• Lemma: In terms of the BKL generators, \( B_n \) is presented by the relations

\[
\begin{align*}
\text{for disjoint chords,} \\
\text{for adjacent chords enumerated in clockwise order.}
\end{align*}
\]

Hence: For \( P \) a \( p \)-gon, can define \( a_P \) to be the product of the \( a_{i,j} \) corresponding to \( p-1 \) adjacent edges of \( P \) in clockwise order; idem for an union of disjoint polygons.
• Chord representation of the Birman–Ko–Lee generators: \( a_{i,j} \mapsto \)

• **Lemma:** In terms of the BKL generators, \( B_n \) is presented by the relations

\[
\begin{align*}
\cdot &= \cdot \quad \text{for disjoint chords,} \\
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\]

Hence: For \( P \) a \( p \)-gon, can define \( a_P \) to be the product of the \( a_{i,j} \) corresponding to \( p-1 \) adjacent edges of \( P \) in clockwise order; *idem* for an union of disjoint polygons.

• **Proposition** (Digne–Michel 2002): The divisors of \( \Delta_n^* \) in \( B_n^{++} \) are the \( \frac{1}{n+1} \binom{2n}{n} \) elements \( a_P \) for \( P \) a non-intersecting union of polygons in an \( n \)-punctured circle.
Chords

- Chord representation of the Birman–Ko–Lee generators: \( a_{i,j} \rightarrow \)

- **Lemma:** In terms of the BKL generators, \( B_n \) is presented by the relations
  - for disjoint chords,
  - for adjacent chords enumerated in clockwise order.

  Hence: For \( P \) a \( p \)-gon, can define \( a_P \) to be the product of the \( a_{i,j} \) corresponding to \( p-1 \) adjacent edges of \( P \) in clockwise order; \( \text{idem} \) for an union of disjoint polygons.

- **Proposition** (Digne–Michel 2002): The divisors of \( \Delta_n^* \) in \( B_n^{++} \) are the \( \frac{1}{n+1} \binom{2n}{n} \) elements \( a_P \) for \( P \) a non-intersecting union of polygons in an \( n \)-punctured circle.

  equivalently: a non-crossing partition of \( \{1, \ldots, n\} \)
Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
- 3. Dynnikov’s coordinates
- 4. Bressaud’s relaxation algorithm
• An $n$-strand braid diagram = a danse of $n$ points in a disk:
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... an isotopy class of homeomorphisms of $D_n$ leaving $\partial D_n$ fixed
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... ➤ an isotopy class of homeomorphisms of $D_n$ leaving $\partial D_n$ fixed
disk with $n$ marked points
• An $n$-strand braid diagram = a danse of $n$ points in a disk:

... ► an isotopy class of homeomorphisms of $D_n$ leaving $\partial D_n$ fixed

disk with $n$ marked points           boundary of $D_n$
• An $n$-strand braid diagram = a danse of $n$ points in a disk:

... $\uparrow$ an isotopy class of homeomorphisms of $D_n$ leaving $\partial D_n$ fixed

disk with $n$ marked points $\uparrow$ boundary of $D_n$

• **Proposition:** The group $B_n$ is (isomorphic to) the mapping class group of $D_n$. 
• Viewing $B_n$ as a group of *(isotopy classes of)* homeomorphisms of $D_n$: 
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  ► action of $B_n$ on the fundamental group of $D_n$, 
• Viewing \( B_n \) as a group of (isotopy classes of) homeomorphisms of \( D_n \):
  
  - action of \( B_n \) on the fundamental group of \( D_n \), a free group of rank \( n \).
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  ► action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$. 

\[ \begin{align*}
  & D_3 \\
  & \bullet \\
  & \ast \\
  & x_1 
\end{align*} \]
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  ▶ action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$.  

![Diagram of $D_3$ with points $x_1$, $x_2$, and a star (*) indicating action.]
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  - action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$. 

![Diagram of $D_3$ with points labeled $x_1$, $x_2$, and $x_3$]
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  ▶ action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$. 

![Diagram of $D_3$ with points $x_1$, $x_2$, $x_3$, and $\sigma_1$]
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  ▶ action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$. 

![Diagram of $D_3$ and action of $B_n$]
The Artin representation

- Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  - action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$.
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• From there: a homomorphism $\rho$ from $B_n$ to $\text{Aut}(F_n)$:
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
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  \[
  \rho(\sigma_i) : \begin{cases} 
  x_i \mapsto x_i x_{i+1} x_i^{-1}, 
  \end{cases}
  \]
The Artin representation

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• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  - action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$.

  ![Diagram showing action of $B_n$ on $D_3$]

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  $$
  \rho(\sigma_i) : \left\{ \begin{array}{ll}
  x_i & \mapsto x_i x_{i+1} x_i^{-1}, \\
  x_{i+1} & \mapsto x_i, \\
  x_k & \mapsto x_k \text{ for } k \neq i, i + 1.
  \end{array} \right.
  $$
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
  - action of $B_n$ on the fundamental group of $D_n$, a free group of rank $n$.

$$
\bullet \quad \sigma_1
$$

• From there: a homomorphism $\rho$ from $B_n$ to $\text{Aut}(F_n)$:
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• **Theorem (Artin):** The homomorphism $\rho$ is injective.
• Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
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• From there: a homomorphism $\rho$ from $B_n$ to $\text{Aut}(F_n)$:
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  \rho(\sigma_i) : \begin{cases} 
  x_i &\mapsto x_ix_{i+1}x_i^{-1}, \\
  x_{i+1} &\mapsto x_i, \\
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  \end{cases}
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• **Theorem (Artin):** The homomorphism $\rho$ is injective.

  ▶ a new solution of the Word Problem for $B_n$ (hence of the Braid Isotopy Problem):
The Artin representation

- Viewing $B_n$ as a group of (isotopy classes of) homeomorphisms of $D_n$:
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- From there: a homomorphism $\rho$ from $B_n$ to $\text{Aut}(F_n)$:
  
  \[
  \begin{align*}
  \rho(\sigma_i) : & \quad 
  \begin{cases}
  x_i \mapsto x_i x_{i+1} x_i^{-1}, \\
  x_{i+1} \mapsto x_i, \\
  x_k \mapsto x_k \text{ for } k \neq i, i+1.
  \end{cases}
  \end{align*}
  \]

- **Theorem** (Artin): *The homomorphism $\rho$ is injective.*

  - a new solution of the Word Problem for $B_n$ (hence of the Braid Isotopy Problem):
    a braid word $w$ represents 1 in $B_n$ iff $\rho(w)(x_k) = x_k$ holds for $k = 1, \ldots, n$. 
• For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$,
For $x \in \mathbb{Z}$, put $x^+ = \max(0, x), \ x^- = \min(x, 0)$, and

$$F^+(x_1, y_1, x_2, y_2) =$$
• For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

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For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
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with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$. 
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  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

• Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by

  $$(a_1, b_1, ..., a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, ..., a'_n, b'_n)$$
• For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

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• Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by

$$(a_1, b_1, ..., a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, ..., a'_n, b'_n)$$

with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and
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  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

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  \]
  with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and $(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1})$. 
• For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
  
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  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

• Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by
  
  $$(a_1, b_1, ..., a_n, b_n) * \sigma_i^e = (a_1', b_1', ..., a_n', b_n')$$
  
  with $a_k' = a_k$ and $b_k' = b_k$ for $k \neq i, i+1$, and
  
  $$(a_i', b_i', a_{i+1}', b_{i+1}') = F^e(a_i, b_i, a_{i+1}, b_{i+1})$$

• Definition: The coordinates of an $n$-strand braid word $w$ are $(0, 1, 0, 1, ..., 0, 1) * w$. 
Dynnikov coordinates

- For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
  \[
  F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1^+), y_2 - z_1^-),
  \]
  \[
  F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^+ - (y_2^+ + z_2^-), y_2 + z_2^-),
  \]
  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

- Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by
  \[
  (a_1, b_1, ..., a_n, b_n) * \sigma_i^e = (a'_1, b'_1, ..., a'_n, b'_n)
  \]
  with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and $(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1})$.

- **Definition**: The coordinates of an $n$-strand braid word $w$ are $(0, 1, 0, 1, ..., 0, 1) * w$.

- **Theorem** (Dynnikov 2000): The coordinates of $w$ only depend on the braid represented by $w$,
For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

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F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1^+), y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1^-), y_1 + z_1^+),
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with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

- Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by

\[
(a_1, b_1, \ldots, a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, \ldots, a'_n, b'_n)
\]

with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and $(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1})$.

- Definition: The coordinates of an $n$-strand braid word $w$ are $(0, 1, 0, 1, \ldots, 0, 1) \ast w$.

- Theorem (Dynnikov 2000): The coordinates of $w$ only depend on the braid represented by $w$, and they characterize the latter.
• For \( x \in \mathbb{Z} \), put \( x^+ = \max(0, x) \), \( x^- = \min(x, 0) \), and
\[
F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1)^+, y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1)^-, y_1 + z_1^+),
\]
\[
F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^+ - (y_2^+ + z_2)^+, y_2 + z_2^-, x_2 - y_2^- - (y_1^- - z_2)^-, y_1 - z_2^-),
\]
with \( z_1 = x_1 - y_1^- - x_2 + y_2^+ \) and \( z_2 = x_1 + y_1^- - x_2 - y_2^+ \).

• Define an action of \( n \)-strand braid words on \( \mathbb{Z}^{2n} \) by
\[
(a_1, b_1, \ldots, a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, \ldots, a'_n, b'_n)
\]
with \( a'_k = a_k \) and \( b'_k = b_k \) for \( k \neq i, i+1 \), and \((a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1})\).

• Definition: The coordinates of an \( n \)-strand braid word \( w \) are \((0, 1, 0, 1, \ldots, 0, 1) \ast w\).

• Theorem (Dynnikov 2000): The coordinates of \( w \) only depend on the braid represented by \( w \), and they characterize the latter.

▷ Hence: a new solution of the Braid Isotopy Problem:
• For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
  
  $F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1^+), y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1^-), \ y_1 + z_1^+),$ 
  
  $F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^- - (y_2^+ + z_2^-), y_2 + z_2^-, x_2 - y_2^- - (y_1^- - z_2^-), \ y_1 - z_2^-),$ 

  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

• Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by
  
  $(a_1, b_1, ..., a_n, b_n) * \sigma_i^e = (a'_1, b'_1, ..., a'_n, b'_n)$

  with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and $(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1}).$

• Definition: The coordinates of an $n$-strand braid word $w$ are $(0, 1, 0, 1, ..., 0, 1) * w$.

• Theorem (Dynnikov 2000): The coordinates of $w$ only depend on the braid represented by $w$, and they characterize the latter.

  ▶ Hence: a new solution of the Braid Isotopy Problem:
  
  a braid word $w$ represents 1 iff its Dynnikov coordinates are $(0, 1, 0, 1, ..., 0, 1)$. 

Dynnikov coordinates
Dynnikov coordinates

- For $x \in \mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and
  \[
  F^+(x_1, y_1, x_2, y_2) = (x_1 + y_1^+ + (y_2^+ - z_1)^+, y_2 - z_1^+, x_2 + y_2^- + (y_1^- + z_1)^-, y_1 + z_1^+),
  \]
  \[
  F^-(x_1, y_1, x_2, y_2) = (x_1 - y_1^- - (y_2^+ + z_2)^+, y_2 + z_2^-, x_2 - y_2^- - (y_1^- - z_2)^-, y_1 - z_2^-),
  \]
  with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

- Define an action of $n$-strand braid words on $\mathbb{Z}^{2n}$ by
  \[
  (a_1, b_1, \ldots, a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, \ldots, a'_n, b'_n)
  \]
  with $a'_k = a_k$ and $b'_k = b_k$ for $k \neq i, i+1$, and
  \[
  (a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1}).
  \]

- Definition: The coordinates of an $n$-strand braid word $w$ are $(0, 1, 0, 1, \ldots, 0, 1) \ast w$.

- Theorem (Dynnikov 2000): The coordinates of $w$ only depend on the braid represented by $w$, and they characterize the latter.

  - Hence: a new solution of the Braid Isotopy Problem: a braid word $w$ represents 1 iff its Dynnikov coordinates are $(0, 1, 0, 1, \ldots, 0, 1)$.
  - An extremely efficient method: “linear space, quadratic time complexity”
• Braid-homeomorphism of $D_n$ acts on curves drawn in $D_n$. 
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• Braid homeomorphism of $D_n$ acts on curves drawn in $D_n$. 
• Braid = homeomorphism of $D_n$ acts on curves drawn in $D_n$. 
• Braid-homeomorphism of $D_n$ acts on curves drawn in $D_n$. 

\[ \sigma_1 \rightarrow \]
• Braid=homeomorphism of $D_n$ acts on curves drawn in $D_n$. 

\[ \sigma_1 \to \sigma_{2^{-1}} \to \]
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\[ \sigma_1 \rightarrow \sigma_1 \rightarrow \sigma_1 \rightarrow \sigma_1 \rightarrow \sigma_2^{-1} \rightarrow \]
• Braid-homeomorphism of $D_n$ acts on curves drawn in $D_n$. 
• Braid=$\text{homeomorphism of } D_n$ acts on curves drawn in $D_n$.

• Count intersections with a fixed triangulation:
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- Count intersections with a fixed triangulation:

  $3n + 3$ numbers, which determine the braid
Updating coordinates
• **Fact:** The Dynnikov coordinates are the half-differences between the previous intersection numbers. (going from $3n + 3$ downto $2n$)
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  - compare the intersections of $L$ and $\sigma_i(L)$ with the (fixed) triangulation $T$
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$$\#(\sigma_i(L) \cap T) = \#(L \cap \sigma_i^{-1}(T)).$$
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• **Lemma**: If $T, T'$ are any two (singular) triangulations, one can go from $T$ to $T'$ using a finite sequence of flips.
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• **Lemma:** If $T$, $T'$ are any two (singular) triangulations,
  one can go from $T$ to $T'$ using a finite sequence of flips.
• Hence: One must go from $T$ to $\sigma_i^{-1}(T)$ by a finite sequence of flips.
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&x_1 x_2 x_3 x_4 \\
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\[
x + x' =
\]
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- Dynnikov’s formulas when iterating four times (four flips).
Plan:

- 1. The braid isotopy problem
- 2. Greedy normal form and the Garside structure
- 3. Dynnikov’s coordinates
- 4. Bressaud’s relaxation algorithm
• Here again: $n$-strand braid = (isotopy class of) homeomorphism of $D_n$
• Here again: $n$-strand braid $= (\text{isotopy class of})$ homeomorphism of $D_n$

• **Principle**: Fix one (or several) base curve $C$, 
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• **Exemple** (Fenn et al. 1997, Dynnikov–Wiest 2006):
  $C =$ main diameter of $D_n$, strategy $=$ consider the “useful arc”.
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![Diagram showing a circle with points marked, representing the main diameter of $D_n$.](image-url)
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\[
\begin{align*}
\sigma_2^{-1} \sigma_1^{-1} & \quad \rightarrow \\
\end{align*}
\]
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![Diagrams showing the process of unbraiding a braid with specific operations and notation.](image-url)
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\[
\sigma_2^{-1} \sigma_1^{-1} \rightarrow \sigma_2 \sigma_1^{-1} \rightarrow \sigma_2^{-1} \rightarrow \sigma_2^{-1}
\]

whence \( \beta = \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \)
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  ![Diagram](image)
• **Exemple 2** (Bressaud 2005):
  - here $C = \text{axes of standard loops}$
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  - here $C = \text{axes of standard loops}$
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![Diagram showing the reduction of intersections with half-axes](image-url)
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\[
\sigma_2 \rightarrow \sigma_1 \sigma_2^{-1} \rightarrow \sigma_1 \sigma_2
\]
• **Exemple 2** (Bressaud 2005):
  
  ▶ here \( C = \text{axes of standard loops} \)
  
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  ![Diagram showing the process from $\sigma_2$ to $\sigma_1 \sigma_2^{-1}$ to $\sigma_1 \sigma_2$]

  ▶ a normal form on $B_n$ (whence a solution to the Braid Isotopy Problem),
• **Exemple 2** (Bressaud 2005):
  
  - here $C = \text{axes of standard loops}$
  
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  ![Diagram showing the transformation of base loops](image)

  - a normal form on $B_n$ (whence a solution to the Braid Isotopy Problem),
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```
\[
\begin{align*}
\sigma_2 & \rightarrow \sigma_1 \sigma_2^{-1} & \rightarrow \sigma_1 \sigma_2
\end{align*}
\]
```

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• **Remark:** The Bressaud normal form has nothing to do with positive braids and $B^+_n$. 
• **Exemple 2** (Bressaud 2005):
  
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  \[
  \sigma_2 \rightarrow \sigma_1 \sigma_2^{-1} \rightarrow \sigma_1 \sigma_2
  \]

  - a normal form on $B_n$ (whence a solution to the Braid Isotopy Problem),
  - together with an **algorithm** computing $\text{NF}(w \sigma_i^{\pm 1})$ from $\text{NF}(w)$ and $i$.

• **Remark**: The Bressaud normal form has nothing to do with positive braids and $B_{n+}$ (nor with $B_{n+}^*$ either).
\( \sigma_1 : \)

\[
\begin{bmatrix}
1 & 2 & 3 & 4
\end{bmatrix}
\]
Bressaud's algorithm

\[ \sigma_1 : \]

\[ \sigma_2 : \]
Bressaud’s algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
Bressaud's algorithm

1 2 3 4

$\sigma_1$:

$\sigma_2$:

$\sigma_3$:

$\sigma_1^{-1}$:
Bressaud’s algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
Bressaud's algorithm

\[ \sigma_1 : \]

\[ \sigma_2 : \]

\[ \sigma_3 : \]

\[ \sigma_1^{-1} : \]

\[ \sigma_2^{-1} : \]

\[ \sigma_3^{-1} : \]

\[ \sigma_1^{-1} \sigma_2^{-1} : \]
Bressaud's algorithm

\[ \sigma_1 : \]

\[ \sigma_2 : \]

\[ \sigma_3 : \]

\[ \sigma_1^{-1} : \]

\[ \sigma_2^{-1} : \]

\[ \sigma_3^{-1} : \]

\[ \sigma_1^{-1} \sigma_2^{-1} : \]

\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

• Normal form of \( \varepsilon \) \( = \varepsilon \).
Bressaud’s algorithm

\[ \sigma_1 : \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ \sigma_2 : \quad \]

\[ \sigma_3 : \quad \]

\[ \sigma_1^{-1} : \quad \]

\[ \sigma_2^{-1} : \quad \]

\[ \sigma_3^{-1} : \quad \]

\[ \sigma_1^{-1} \sigma_2^{-1} : \quad \]

\[ \sigma_2 \sigma_3 : \quad \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces:
6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \)

\[ = \sigma_1^{-1}. \]
Bressaud's algorithm

\[
\begin{align*}
\sigma_1 : & & \quad \quad \quad \quad & 1 & 2 & 3 & 4 \\
\sigma_2 : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_3 : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_1^{-1} : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_2^{-1} : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_3^{-1} : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_1^{-1} \sigma_2^{-1} : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\sigma_2 \sigma_3 : & & \quad \quad \quad \quad & & \quad \quad & \quad \quad & \quad \quad \\
\end{align*}
\]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of $\sigma_1^{-1} \sigma_2^{-1}$
Bressaud’s algorithm

• Normal form of $\sigma_1^{-1} \sigma_2^{-1}$

etc. (12 pieces: 6 positive, 6 negative)

$\sigma_1^{-1} \sigma_2^{-1} = \sigma_1^{-1} \sigma_2^{-1}$. 
Bressaud's algorithm

\[
\begin{align*}
\sigma_1 : & \quad \sigma_2 : \\
\sigma_3 : & \quad \sigma_1^{-1} : \\
\sigma_2^{-1} : & \quad \sigma_3^{-1} : \\
\sigma_1^{-1} \sigma_2^{-1} : & \\
\sigma_2 \sigma_3 : & \\
\end{align*}
\]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \)
Bressaud’s algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \)

\[ = \sigma_2 \sigma_1^{-1} \sigma_2^{-1}. \]
Bressaud’s algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \)
Bressaud’s algorithm

- \( \sigma_1 \): 1 2 3 4
- \( \sigma_2 \):
- \( \sigma_3 \):
- \( \sigma_1^{-1} \):
- \( \sigma_2^{-1} \):
- \( \sigma_3^{-1} \):
- \( \sigma_1^{-1} \sigma_2^{-1} \):
- \( \sigma_2 \sigma_3 \):

\[ \text{etc. (12 pieces: 6 positive, 6 negative)} \]

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \)
Bressaud’s algorithm

\[
\begin{align*}
\sigma_1 : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_2 : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_3 : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_1^{-1} : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_2^{-1} : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_3^{-1} : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_1^{-1}\sigma_2^{-1} : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\sigma_2\sigma_3 : & \quad 1 \quad 2 \quad 3 \quad 4 \\
\end{align*}
\]

etc. (12 pieces: 6 positive, 6 negative)

• Normal form of \( \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1 \) = \( \sigma_2\sigma_2\sigma_1^{-1}\sigma_2^{-1} \).
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \)
Bressaud's algorithm

\[ \sigma_1 : \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ \sigma_1^{-1} : \quad \text{etc.} \]

\[ \sigma_2 : \quad \text{etc.} \]

\[ \sigma_3 : \quad \text{etc.} \]

\[ \sigma_2^{-1} : \quad \text{etc.} \]

\[ \sigma_3^{-1} : \quad \text{etc.} \]

\[ \sigma_1^{-1} \sigma_2^{-1} : \quad \text{etc.} \]

\[ \sigma_2 \sigma_3 : \quad \text{etc.} \]

(12 pieces: 6 positive, 6 negative)

- **Normal form of** \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \)

\[ = \sigma_2 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1}. \]
Bressaud's algorithm

\[ \sigma_1 : \quad \sigma_2 : \quad \sigma_3 : \]
\[ \sigma_1^{-1} : \quad \sigma_2^{-1} : \quad \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \quad \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \)
Bressaud’s algorithm

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1$ :</td>
<td><img src="Diagram1.png" alt="Diagram" /></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_2$ :</td>
<td></td>
<td><img src="Diagram2.png" alt="Diagram" /></td>
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<tr>
<td>$\sigma_3$ :</td>
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<td><img src="Diagram3.png" alt="Diagram" /></td>
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<tr>
<td>$\sigma_1^{-1}$ :</td>
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<td><img src="Diagram4.png" alt="Diagram" /></td>
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<tr>
<td>$\sigma_2^{-1}$ :</td>
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<td></td>
<td><img src="Diagram5.png" alt="Diagram" /></td>
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</tr>
<tr>
<td>$\sigma_3^{-1}$ :</td>
<td></td>
<td><img src="Diagram6.png" alt="Diagram" /></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_1^{-1}\sigma_2^{-1}$ :</td>
<td><img src="Diagram7.png" alt="Diagram" /></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_2\sigma_3$ :</td>
<td></td>
<td><img src="Diagram8.png" alt="Diagram" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_3^{-1}\sigma_1^{-1}$
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \)
Bressaud’s algorithm

\[
\begin{align*}
\sigma_1 : & \quad 1 \\
\sigma_2 : & \quad 2 \\
\sigma_3 : & \quad 3 \\
\sigma_1^{-1} : & \quad 4 \\
\sigma_1^{-1} \sigma_2^{-1} : & \quad 1, 2, 3, 4 \\
\sigma_2^{-1} : & \quad 1, 2, 3, 4 \\
\sigma_3^{-1} : & \quad 1, 2, 3, 4 \\
\sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} : & \quad 1, 2, 3, 4 \\
\sigma_2 \sigma_3 : & \quad 1, 2, 3, 4 \\
\text{etc. (12 pieces: } & \\
& \text{6 positive, 6 negative)} \\
\text{Normal form of } & \\
& \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_1^{-1} = \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1}.
\end{align*}
\]
Bressaud's algorithm

1 2 3 4

\(\sigma_1:\)
\(\sigma_2:\)
\(\sigma_3:\)
\(\sigma_1^{-1}:\)
\(\sigma_2^{-1}:\)
\(\sigma_3^{-1}:\)
\(\sigma_1^{-1}\sigma_2^{-1}:\)
\(\sigma_2\sigma_3:\)

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of \(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces:
6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \) = \( \sigma_2 \sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \).
Bressaud's algorithm

\[ \begin{align*}
  \sigma_1 &: \\
  \sigma_2 &: \\
  \sigma_3 &: \\
  \sigma_1^{-1} &: \\
  \sigma_2^{-1} &: \\
  \sigma_3^{-1} &: \\
  \sigma_1^{-1} \sigma_2^{-1} &: \\
  \sigma_2 \sigma_3 &:
\end{align*} \]

etc. (12 pieces:
  6 positive, 6 negative)

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \)
Bressaud’s algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \) = \( \sigma_2 \sigma_3 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \).
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sigma_3 \)
Bressaud's algorithm

\[
\begin{align*}
\sigma_1 &: \\
\sigma_2 &: \\
\sigma_3 &: \\
\sigma_1^{-1} &: \\
\sigma_2^{-1} &: \\
\sigma_3^{-1} &: \\
\sigma_1^{-1}\sigma_2^{-1} &: \\
\sigma_2\sigma_3 &:
\end{align*}
\]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3\)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sigma_3 \)
**Bressaud's algorithm**

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- **Normal** form of \( \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sigma_3 \)
Bressaud's algorithm

\[ \sigma_1 : \]
\[ \sigma_2 : \]
\[ \sigma_3 : \]
\[ \sigma_1^{-1} : \]
\[ \sigma_2^{-1} : \]
\[ \sigma_3^{-1} : \]
\[ \sigma_1^{-1} \sigma_2^{-1} : \]
\[ \sigma_2 \sigma_3 : \]

etc. (12 pieces: 6 positive, 6 negative)

- Normal form of \[ \sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sigma_3 = \sigma_2 \sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1}. \]
Bressaud’s algorithm

\[\sigma_1 : \]

\[\sigma_2 : \]

\[\sigma_3 : \]

\[\sigma_1^{-1} : \]

\[\sigma_2^{-1} : \]

\[\sigma_3^{-1} : \]

\[\sigma_1^{-1}\sigma_2^{-1} : \]

\[\sigma_2\sigma_3 : \]

e tc. (12 pieces:
   6 positive, 6 negative)

• Normal form of \(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_3\) \(= \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\).
On the Garside approach:

On the Garside approach:


On the Garside approach:


On the Dynnikov coordinates:

On the Garside approach:


On the Dynnikov coordinates:


On relaxation methods:

On the Garside approach:


On the Dynnikov coordinates:


On relaxation methods:


On the Garside approach:


- **P. Dehornoy**, with **F. Digne, D. Krammer, J. Michel**, Foundations of Garside Theory,

On the Dynnikov coordinates:

- **P. Dehornoy**, with **I. Dynnikov, D. Rolfsen, B. Wiest**, Ordering braids,

On relaxation methods:

- **R. Fenn, M.T. Greene, D. Rolfsen, C. Rourke, B. Wiest**, *Ordering the braid groups*,


[www.math.unicaen.fr/~dehornoy](http://www.math.unicaen.fr/~dehornoy)