Braid Order: History and Connection with Knots

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- An introduction to some of the many aspects of the standard braid order, with an emphasis on the few known connections with knot theory.
Plan:

- The Braid Order in Antiquity
- The Braid Order in the Middle Ages
- The Braid Order in Modern Times (Knot Applications)
I. The Braid Order in Antiquity : 1985-95

- The set-theoretical roots
Artin's braid group $B_n$: $\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| \geq 2, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| = 1 \rangle$.

$\simeq \{ \text{braid diagrams} \}/ \text{isotopy}$:

$\sigma_i \leftrightarrow 1 \ 2 \ \ldots \ i \ i+1 \ \ldots \ n$

$\simeq$ mapping class group of $D_n$ (disk with $n$ punctures):

$\sigma_i \leftrightarrow \begin{array}{c}
1 \\
\cdot \\
\cdot \\
\vdots \\
i \\
\cdot \\
\cdot \\
i+1 \\
\cdot \\
\cdot \\
\ldots \\
\cdot \\
n
\end{array}$
The standard braid order

- A $\sigma$-positive braid diagram:

- **Theorem 1** (D., 1992): For $\beta, \beta'$ in $B_n$, declare $\beta < \beta'$ if $\beta^{-1}\beta'$ can be represented by a $\sigma$-positive diagram. Then $<$ is a left-invariant linear ordering on $B_n$.

  $\beta < \beta'$ implies $\alpha \beta < \alpha \beta'$

- Example: $\beta = \sigma_1$, $\beta' = \sigma_2 \sigma_1$. Then $\beta^{-1}\beta' = \sigma_1^{-1} \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2^{-1}$, so $\beta < \beta'$.

- **Question**: Where does Theorem 1 come from?

- **Theorem 0** (D., 1986): If $j$ is an elementary embedding of a self-similar rank, then the LD-structure of $\text{Iter}(j)$ implies $\Pi^1_1$-determinacy.
• **Braid diagram colorings**: Start with a set $S$ (“colours”), apply colours at the left ends of the strands in a braid diagram, propagate the colors to the right, and compare the initial and final colors.

• **Option 1**: Colors are preserved in crossings:

  $\begin{align*}
  y \times x & \Rightarrow \text{permutation of colors} \\
  x \times y & \\
  x \times x \ast y & 
  \end{align*}$

• **Option 2**: (Joyce, Matveev, Brieskorn, ...) Colors change under the rule

  $\begin{align*}
  y \times x & \Rightarrow \text{permutation of colors} \\
  x \times x \ast y & \Rightarrow \text{where } \ast \text{ is some binary operation on } S.
  \end{align*}$

• For an action of $B_n$ on $S^n$, one needs **compatibility** with the braid relations:

  $\begin{align*}
  z \times y \times x & \Rightarrow \text{compatibility with the braid relations} \\
  x \ast (y \ast z) & \Rightarrow \text{compatibility with the braid relations} \\
  (x \ast y) \ast (x \ast z) & \Rightarrow \text{compatibility with the braid relations}
  \end{align*}$

• **Hence**: action of $B_n$ iff $\ast$ satisfies the **left self-distributivity** law (LD):

  $$x \ast (y \ast z) = (x \ast y) \ast (x \ast z).$$
• Classical examples of **LD-systems**:

  (= sets equipped with an operation satisfying the LD law)

- \( x \ast y = y \), leads to \( B_n \rightarrow \mathfrak{S}_n \).
- \( x \ast y = xyx^{-1} \), leads to \( B_n \rightarrow \text{Aut}(F_n) \) (Artin)
- \( x \ast y = (1-t)x + ty \), leads to \( B_n \rightarrow \text{GL}_n(\mathbb{Z}[t, t^{-1}]) \) (Burau)

• Note: in these examples, \( x \ast x = x \) always holds.

**Definition**: Say that an LD-system \((S, \ast)\) is **orderable** if there is a linear ordering \(<\) on \(S\) satisfying \( x < x \ast y \) for all \( x, y \).

• Certainly orderable LD-systems are of a new flavour: \( x < x \ast x \neq x \).

**Theorem 0.5** (D., 1991): There exist orderable LD-systems (namely: free LD-systems).

• **Claim**: Theorem 1 (braid order) directly comes from Theorem 0.5.
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• **Ingredient 1:** A $\sigma$-positive braid never represents 1.

\[
\begin{align*}
x & < x \cdot y_1 < (x \cdot y_1) \cdot y_2 < \ldots \\
x & \neq x
\end{align*}
\]

• **Ingredient 2:** Every two braids are comparable.

\[
\begin{align*}
&x_3 \quad y_3 \\
x_2 \quad y_2 \\
x_1 \quad y_1 \\
\end{align*} \implies \begin{align*}
&(x_3, y_3, z_3) \\
&(x_2, y_2, z_2) \\
&(x_1, y_1, z_1)
\end{align*}
\]

Compare $(y_1, y_2, \ldots)$ and $(z_1, z_2, \ldots)$ lexicographically.

• **Question:** OK, but then, why to look for orderable LD-systems?
... because set theory told us

- Set theory studies infinity. By Gödel’s theorem, every axiomatic system, e.g., the standard Zermelo-Fraenkel system $ZF$, is incomplete.

- Gödel’s program: Complete ZF with axioms stating the existence of “hyper-infinite” sets (“large cardinals”).

- Typically, strengthen

  “$X$ is infinite iff $\exists j : X \to X$ injective non-surjective”

into

“$X$ is self-similar (= hyper-infinite) iff $\exists j : X \to X$ injective non-surjective and, moreover, $j$ preserves everything that is definable from $\in$. An elementary embedding

- Example: As $j : n \mapsto n + 1$ is injective non-surjective, $\mathbb{N}$ is infinite; Now $j$ preserves $<$, but not $+$: $j$ is not an elementary embedding. In fact: no (non-trivial) elementary embedding of $\mathbb{N}$ exists: $\mathbb{N}$ is infinite, but not self-similar.
**Definition:** A rank is a set $R$ such that $f : R \to R$ implies $f \in R$. (??)

- If $R$ is a self-similar rank (i.e., there exists an element embedding of $R$ into itself) and $i, j$ are elementary embeddings of $R$, then we can apply $i$ to $j$.
- “Being an elem. emb.” is definable from $\in$, so $i(j)$ is an elementary embedding: “application” is a binary operation on elementary embeddings of $R$.
- “Being the image under” is definable from $\in$, so $\ell = j(k)$ implies $i(\ell) = i(j)(i(k))$, $i(j(k)) = i(j)(i(k))$:
  
  the “application” operation satisfies the LD law.

**Proposition:** If $j$ is an elementary embedding of a self-similar rank, then the iterates of $j$, make an LD-system $\text{Iter}(j)$.

Closure of $\{j\}$ under application: $j(j), j(j)(j), \ldots$
Remember the question: why to look for orderable LD-systems (Theorem 0.5)?

- **Theorem 0.1** (D., 1989): If there exists at least one orderable LD-system, then the word problem of LD is solvable.

- **Theorem 0.2** (Laver, 1989): If $j$ is an elem. embedding of a self-similar rank, then $\text{Iter}(j)$ is an orderable LD-system.

- **Corollary**: If there exists a self-similar rank, the word problem of LD is solvable.

- **But** the existence of a self-similar rank is an unprovable axiom, so the corollary does not give a solution for the word problem of LD.

  - Construct a **true** orderable LD-system: Theorem 0.5 (orderable LD-systems) by investigating the “geometry group of LD”.
  - As the latter group extends Artin’s braid group: Theorem 1 (braid order).
An application of set theory?

- **Question**: Why care about \( \text{Iter}(j) \) and prove Theorems 0.1 and 0.2?

- **Theorem 0 (D., 1986)**: If \( j \) is an elem. embedding of a self-similar rank, then the LD-structure of \( \text{Iter}(j) \) implies \( \Pi^1_1 \)-determinacy. ("\( \text{Iter}(j) \) is not trivial")

\[ \leadsto \] a continuous path from Theorem 0 (about sets) to Theorem 1 (about braids).

- *Is the braid order an application of set theory?*
  - Formally, **no**: braids appear when sets disappear.
  - In essence, **yes**: orderable LD-systems have been investigated because set theory showed they might exist and be involved in deep phenomena.

- **An analogy**:
  - In physics: using physical intuition and/or evidence, guess some statement, then pass it to mathematicians for a formal proof.
  - Here: using logical intuition and/or evidence (\( \exists \) self-similar rank), guess some statement (\( \exists \) orderable LD-system), then pass it to mathematicians for a formal proof.
II. The Braid Order in the Middle Ages: 1995-2000

- Handle reduction
- Dynnikov’s formulas
Many different approaches

- **Theorems** (Burckel, D., Dynnikov, Fenn, Fomenko, Funk, Greene, Larue, Rolfsen, Rourke, Short, Wiest, ...):
  “Many different approaches lead to the same braid order”.

- **Theorems** (Clay, Ito, Navas, Rolfsen, Short, Wiest, ...):
  “There exist many different braid orders making an interesting space”.
Handle reduction

- A $\sigma_i$-handle:

- Reducing a handle:

- Handle reduction is an isotopy; It extends free group reduction;
Words with no handle are: the empty word, $\sigma$-positive words, $\sigma$-negative words.

- Theorem (D., 1995): A braid $\beta$ satisfies $\beta = 1$ (resp. $\beta > 1$) iff some/any sequence of handle reductions from some/any braid word representing $\beta$ finishes with the empty word (resp. with a $\sigma$-positive word).
Dynnikov’s formulas

• **Definition**: For $x$ in $\mathbb{Z}$, put $x^+ = \max(0, x)$, $x^- = \min(x, 0)$, and

$$F^+(x_1, y_1, x_2, y_2) = 
(x_1 + y_1^+ + (y_2^+ - z_1)^+, y_2 - z_1^+, x_2 + y_2^- + (y_1 + z_1^-), y_1 + z_1^+),$$

$$F^-(x_1, y_1, x_2, y_2) = 
(x_1 - y_1^+ - (y_2 + z_2)^+, y_2 + z_2^-, x_2 - y_2^- - (y_1 - z_2^-), y_1 - z_2^-),$$

with $z_1 = x_1 - y_1^- - x_2 + y_2^+$ and $z_2 = x_1 + y_1^- - x_2 - y_2^+$.

• **Let** $n$-strand braid words act on $\mathbb{Z}^{2n}$ by

$$(a_1, b_1, \ldots, a_n, b_n) \ast \sigma_i^e = (a'_1, b'_1, \ldots, a'_n, b'_n)$$

with $a'_k = a_k$ et $b'_k = b_k$ for $k \neq i, i + 1$, and

$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = F^e(a_i, b_i, a_{i+1}, b_{i+1}).$$

• **Finally**, define the **coordinates** of a braid word $w$ to be $(0, 1, 0, 1, \ldots, 0, 1) \ast w$.

• **Remark**: Looks awkward, but actually very easy to implement.

• **Theorem** (Dynnikov, 2000): A braid $\beta$ satisfies $\beta = 1$ (resp. $\beta > 1$) iff the coordinates of some/any braid word representing $\beta$ are $(0, 1, 0, 1, \ldots, 0, 1)$ (resp. the first nonzero odd rank coordinate is positive).
Braid = homeomorphism acts on laminations of $D_n$ (embedded in $S^2$).

Count intersections of $\beta(L_*)$ with a fixed triangulation $T_*$:

$\sim 3n + 3$ numbers that determine the braid; coordinates = half-differences (going from $3n + 3$ to $2n$ numbers)
• **Question**: What are the coordinates of $\beta \sigma_i$ in terms of those of $\beta$ and of $i$?

\[
\text{compare the intersections of } L \text{ and } \sigma_i(L) \text{ with the base triangulation } T_*. \\
\text{a lamination } \approx \text{ family of closed curve(s)}
\]

• Now $\#(\sigma_i(L) \cap T_*) = \#(L \cap \sigma_i^{-1}(T_*) )$.

Hence: compare the intersections of $L$ with $T_*$ and $\sigma_i^{-1}(T_*)$.

• **Fact**: If $T, T'$ are (singular) triangulations of a surface, one can go from $T$ to $T'$ by a finite sequence of flips:
• For one flip:

\[ x + x' = \max(x_1 + x_3, x_2 + x_4) \]

\[ \Rightarrow \text{Dynnikov’s formulae by a fourfold iteration.} \]
III. The Braid Order in Modern Times: 2000-...

- Floor and closure
- Conjugacy via the $\mu$ function
Definition: For $\beta$ in $B_n$, the floor $\lfloor \beta \rfloor$ is the unique $m$ satisfying
\[ \Delta_n^{2m} \leq \beta < \Delta_n^{2m+2}. \]

Proposition (Malyutin–Netsvetaev, '00): The stable floor $\lfloor \beta \rfloor_s$ of $\beta$, defined to be $\lim_p \lfloor \beta^p \rfloor / p$, is a pseudo-character on $B_n$: one has $\lfloor \beta^p \rfloor_s = p \lfloor \beta \rfloor_s$, and
\[ \left| \lfloor \beta_1 \beta_2 \rfloor_s - \lfloor \beta_1 \rfloor_s - \lfloor \beta_2 \rfloor_s \right| \leq 1. \]

Principle: If $\beta$ is very small or very large in the braid ordering, then the properties of $\hat{\beta}$ can be read from those of $\beta$. 
• **Theorem** (Malyutin–Netsvetaev, '00): If $\beta$ satisfies $\beta < \Delta_n^{-4}$ or $\beta > \Delta_n^4$, then $\widehat{\beta}$ is prime and non-trivial.

Proof: For $\chi$ a pseudo-character on $B_n$ satisfying $\chi|_{B_{n-1}} = 0$, then $|\chi(\beta)| > \text{defect of } (\chi)$ implies that $\widehat{\beta}$ is prime. Apply to $\lfloor \ _{s} \rfloor$. □

• **Theorem** (Malyutin–Netsvetaev, '00): For each $n$, there exists $r(n)$ such that, if $\beta$ in $B_n$ satisfies $\beta < \Delta_n^{-2r(n)}$ or $\beta > \Delta_n^{2r(n)}$, then $\widehat{\beta}$ is represented by a unique conjugacy class in $B_n$.

$\beta' \approx \widehat{\beta}$ implies $\beta'$ conjugated to $\beta$

Proof: For each template move $M$, there exists $r$ s.t. $|\lfloor \beta \rfloor| > r$ implies that $\widehat{\beta}$ is not eligible for $M$.

By the Birman-Menasco MTWS theory, $\exists$ finitely template moves moves for each $n$. □

• *(M.-N., 2000)* $r(3) \leq 3$; *(Matsuda, 2008)* $r(4) \leq 4$; *(Ito, 2009)* $r(3) = 2$.

• Conjecture (Ito): $r(n) \leq n - 1$ for each $n$. 

• **Theorem (Ito, ’08):** If $\beta$ satisfies $\beta < \Delta_n^{-2k-2}$ or $\beta > \Delta_n^{2k+2}$, then
  
  $4 \cdot \text{genus}(\hat{\beta}) > k(n + 2) - 2$.

• **Theorem (Ito, ’08):** If $\beta$ satisfies $\beta \leq \Delta_n^{-4}$ or $\beta \geq \Delta_n^{4}$ and $\hat{\beta}$ is a knot, then
  - $\beta$ is periodic iff $\hat{\beta}$ is a torus knot,
  - $\beta$ is reducible iff $\hat{\beta}$ is a satellite knot,
  - $\beta$ is pseudo-Anosov iff $\hat{\beta}$ is hyperbolic.

• False in general: the trefoil knot is the closure of $\sigma_1^3$ (periodic), $\sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2$ (reducible), and $\sigma_1^3 \sigma_2^{-1}$ (pseudo-Anosov).
Two functions

- **Theorem (Laver, 1995):** For every braid $\beta$ and every $i$, one has $\beta^{-1}\sigma_i\beta > 1$.

- **Corollary:** The restriction of the braid order to $B_n^+$ is a well-ordering.

  the submonoid of $B_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$

  every nonempty subset has a minimal element

- For $\beta$ in $B_n^+$, put

  $\mu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ conjugate to } \beta\}.$

  $\nu(\beta) = \min\{\beta' \in B_n^+ \mid \beta' \text{ Markov equivalent to } \beta\}.$

- Are these definitions useful? Only if the functions can be computed ...

  ... so certainly not until recently.

- **Remark:** The braid order is quite bizarre: not Archimedean, not Conradian, ...

  $\exists \beta, \beta' > 1 \forall p \ (\beta^p < \beta')$  $\exists \beta, \beta' > 1 \forall p \ (\beta < \beta' \beta^p)$
The alternating normal form

- Associate with every braid $\beta$ in $B_n^+$ a finite sequence $(... , \beta_3, \beta_2, \beta_1)$ of braids in $B_{n-1}^+$: the $n$-splitting of $\beta$.

$$\Phi_n(\beta_4) \downarrow \Phi_n(\beta_2) \downarrow \beta_1$$

- Iterate to obtain a unique normal form ( = construct a tree for each braid )

  The order $<$ on $B_n^+$ is a ShortLex-extension of the order $<$ on $B_{n-1}^+$.

  $\beta < \beta'$ holds iff the splitting of $\beta$ is shorter than that of $\beta'$ or they have the same length and the splitting of $\beta$ is lexicographically smaller
• **Definition (Birman-Ko-Lee, 1997):** The dual braid monoid $B_n^+$ is the submonoid of $B_n$ generated by $(a_{i,j})_{1 \leq i < j \leq n}$ with

$$a_{i,j} = \sigma_{j-1}^{-1} \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \ldots \sigma_{j-1}.$$  

• **Theorem (Fromentin, 2008):** The order $<$ on $B_n^+$ is a ShortLex-extension of the order $<$ on $B_{n-1}^+$. 

• Alternative Garside structure for $B_n$, with $\text{Cat}_n$ non-crossing partitions replacing $n!$ permutations.

• Then similar splitting of braids in $B_n^+$ into sequences of braids in $B_{n-1}^+$. 

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Fromentin’s rotating normal form
- **Principle**: With the (alternating and) rotating normal form(s) of braids, we now have a practical way of controlling the braid order.

- Example: For $\beta$ in $B_n^+$, one has $|\beta| \approx 2 \times \text{(length of the splitting of } \beta) - 2$.

- **Conjecture** (D., Fromentin, Gebhardt, 2009): For $\beta$ in $B_3^+$,
  \[ \mu(\beta \Delta_3^2) = \sigma_1 \sigma_2^2 \sigma_1 \cdot \mu(\beta) \cdot \sigma_1^2. \]

  \[ \min\{\beta' \in B_n^+ \mid \beta' \text{ conjugate to } \beta\} \]

  ... more generally: a reasonable hope of computing the function $\mu$.
  (whence possibly solving the conjugacy problem in a completely new way).

- Then let’s dream a little: What about $\nu$?

  similar to $\mu$ with Markov equivalence instead of conjugacy
• P. Dehornoy, I. Dynnikov, D. Rolfsen, B. Wiest, Ordering braids, 

• J. Fromentin, The well-order on dual braid monoids;

• J. Fromentin, Every braid admits a short sigma-definite expression,

• T. Ito, Braid ordering and the geometry of closed braids,
  arXiv math.GT/0805.1447.

• T. Ito, Braid ordering and knot genus,
  arXiv math.GT/0805.2042.

• A. Malyutin, Twist number of (closed) braids,

• A. Malyutin and N. Netsvetaev, Dehornoy’s ordering on the braid group and braid
  moves,

www.math.unicaen.fr/~dehornoy